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On a Splitting Theorem for Riemannian Manifolds

by

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Introduction

The decomposability of a Riemannian manifold has been studied by many mathematicians. The problem is closely related to the reducibility of holonomy groups. In fact, using holonomy groups, de Rham proved a famous decomposition theorem for simply connected complete Riemannian manifolds (see [1]). Without the assumption of simply-connectedness, the problem is rather difficult (cf. [6]). In this case, the condition of decomposability may be truely topological or differential geometric. For example, the splitting theorem of Lawson and Yau says that, for a compact Riemannian manifold M with non-positive curvature, the condition can be thoroughly described by the fundamental group $\pi_1(M)$ (see [4]). On the other hand, in [5], Toponogov proved the following theorem. Every non-compact complete Riemannian manifold M with non-negative curvature is isometric to the Riemannian product $\mathbf{R}^n \times M'$, where $0 \leqq n \leqq \dim M$ and M' contains no "lines." These theorems are fairly different from de Rham's theorem.

In the present paper, we shall prove a splitting theorem similar to Toponogov's theorem. Instead of "lines," we shall use harmonic functions. Let M be a connected Riemannian manifold with metric tensor g. A function f on M is called an *affine function* on M if, for every geodesic $c(t)$ with affine parameter t, there are real constants a and b such that $f(c(t)) = at + b$ for any t. As was mentioned above, every affine function is harmonic (Proposition 2.2). We remark here that affine functions can also be defined on affinely connected manifolds and that they contain some information on the topological structure of affinely connected manifolds (see [2]).

Let $A(M, g)$ be the set of all affine functions on M. Then it is shown that $A(M, g)$ is a finite-dimensional real vector space with $1 \leqq \dim A(M, g) \leqq \dim M - 1$. Now we can state our splitting theorem as follows.

THEOREM 3.2. *Let M be a non-compact, connected and complete Riemannian manifold with metric tensor g and let $n = \dim A(M, g) - 1$. Then there exists a connected Riemannian manifold M' with metric tensor g' such that M is isometric to the Riemannian product $\mathbf{R}^n \times M'$ and $\dim A(M', g') = 1$.*

It should be remarked that M is not assumed to be simply connected. The completeness of M can be replaced by a weaker condition.

In § 1 and § 2, we shall study some basic properties of affine functions. In § 3, we shall prove our splitting theorem. The crucial point of the proof lies in the fact that, for every affine function f on M, grad f, the gradient of f, is a parallel vector field on M.

Throughout this paper, all manifolds and differential geometric objects on them are assumed to be differentiable of class C^{∞}. For brevity's sake, we shall often use the adjective "smooth" instead of "differentiable."

§ 1. Affine functions

Let M be a connected Riemannian manifold with metric tensor g. For a smooth curve $c(t)$ in M, we denote by $\dot{c}(t)$ the tangent vector to the curve at $c(t)$ and by $D\dot{c}(t)/dt$ the covariant derivative of $\dot{c}(t)$ with respect to the Riemannian connection of M. A smooth curve $c(t)$ in M defined on an open interval I is called a *geodesic* if $D\dot{c}(t)/dt=0$ on I. This being the case, the parameter t is called an *affine parameter*. In this paper, all geodesics under consideration are assumed to be parametrized by affine parameter.

A smooth function f on M is called an *affine function* on M if, for every geodesic $c(t)$, there are real constants a and b such that $f(c(t))=at+b$ for all t where defined. This definition does not depend on the choice of an affine parameter t because any other affine parameter t' is given by an affine transformation $t'=ct+d$, where $c\neq 0$ and d are real constants.

PROPOSITION 1.1. *Let f be an affine function on M. If the differential $(df)_x$ of f at some point x of M vanishes, then f is a constant function on M.*

Proof. Let N denote the set of all points $y\in M$ such that $(df)_y=0$. Clearly, N is non-empty and closed in M. Let us take any $y\in N$ and any geodesic $c(t)$ with $c(0)=y$. Then we can put $f(c(t))=at+b$ $(a, b\in \mathbf{R})$. Hence we have

$$a=\frac{d}{dt}f(c(t))\bigg|_{t=0}=(df)_y(\dot{c}(0))=0\,.$$

This means that f is constant on every geodesic starting from the point y. Let U be a convex neighborhood of y (see [3] p. 166). Since every point of U can be joined to y by a geodesic segment, f is constant on U. Thus $U\subset N$ and hence N is open in M. Since M is connected, we can conclude that f is constant on M.

Let $A(M, g)$ denote the set of all affine functions on M. Then it is clear that $A(M, g)$ is a linear subspace of the real vector space of all smooth functions on M. Let $\mathbf{R}$ denote the field of real numbers. Every element of $\mathbf{R}$ can be canonically identified with a constant function on M, so we get the natural inclusion $i\colon \mathbf{R}\to A(M, g)$.

PROPOSITION 1.2. *$A(M, g)$ is finite-dimensional and satisfies*

$$1 \leqq \dim A(M, g) \leqq \dim M - 1 .$$

Morevoer, if M is compact, then $\dim A(M, g) = 1$.

Proof. Let us fix a point x of M. Let F: $A(M, g) \to T_x^*(M)$ denote the linear mapping given by $F(f) = (df)_x$ ($f \in A(M, g)$), where $T_x^*(M)$ is the cotangent space to M at x. Then Proposition 1.1 implies that the sequence

$$0 \longrightarrow \mathbf{R} \xrightarrow{\ i\ } A(M, g) \xrightarrow{\ F\ } T_x^*(M)$$

is exact. This proves the first and second assertions. The last assertion follows easily from Propostion 1.1 and the fact that every smooth function on a compact manifold has a critical point.

PROPOSITION 1.3. *Let $1, f_1, \cdots, f_n$ be elements of $A(M, g)$. Then the following two statements are equivalent*:

1) *$1, f_1, \cdots, f_n$ are linearly independent in $A(M, g)$*;
2) *$df_1, \cdots, df_n$ are linearly independent at each point of M.*

Proof. Suppose 1). Let x be any point of M and assume that $\sum_{i=1}^{n} a_i(df_i)_x = 0$ for real constants $a_1, \cdots, a_n$. Then we have $(d(\sum_{i=1}^{n} a_i f_i))_x = 0$, so by Proposition 1.1 there is a constant b such that $\sum_{i=1}^{n} a_i f_i + b = 0$. Hence we get $a_i = 0$ for all $i = 1, \cdots, n$, which implies 2). The converse is obvious.

Now we set $a(M, g) = \dim A(M, g) - 1$.

PROPOSITION 1.4. *Let $\mathbf{R}^n$ be the n-dimensional Euclidean space with the standard metric ds^2. Then we have a $(\mathbf{R}^n, ds^2) = n$.*

Proof. Let $(x_1, \cdots, x_n)$ be the canonical coordinate system on $\mathbf{R}^n$. Then the coordinate functions $x_1, \cdots, x_n$ belong to $A(\mathbf{R}^n, ds^2)$. Moreover, it follows from Propositions 1.2 and 1.3 that $1, x_1, \cdots, x_n$ form a basis of $A(\mathbf{R}^n, ds^2)$. Hence we get $a(\mathbf{R}^n, ds^2) = n$.

Let M' be another connected Riemannian manifold with metric tensor g'. For a smooth mapping $h: M \to M'$ and a smooth function f on M', we denote by $h^*(f)$ the smooth function on M given by $h^*(f) = f \circ h$. We have immediately the following proposition.

PROPOSITION 1.5 *Let $h: M \to M'$ be a smooth mapping which maps every geodesic $c(t)$ of M into a geodesic $h(c(t))$ of M' (together with its affine parameter). Then $h^*(A(M', g')) \subset A(M, g)$ and $h^*: A(M', g') \to A(M, g)$ is a linear homomorphism. In particular, if h is an isometry of M onto M'. Then h^* is a linear isomorphism of $A(M', g')$ onto $A(M, g)$.*

PROPOSITION 1.6. *Let M_i be a connected Riemannian manifold with metric tensor g_i ($i=1, 2$). Let g_1+g_2 denote the product Riemannian metric on $M_1 \times M_2$. Then we have*

$$a(M_1 \times M_2, g_1+g_2)=a(M_1, g_1)+a(M_2, g_2).$$

Proof. For simplicity, we write $A=A(M_1 \times M_2, g_1+g_2)$ and $A_i=A(M_i, g_i)$, $i=1, 2$. Since the natural projection $p_i: M_1 \times M_2 \to M_i$ maps every geodesic of $M_1 \times M_2$ into a geodesic of M_i, $p_i^*: A_i \to A$ is an injective homomorphism ($i=1, 2$). Let us fix a point (x_0, y_0) of $M_1 \times M_2$ ($x_0 \in M_1$, $y_0 \in M_2$). Let $h_i: M_i \to M_1 \times M_2$, $i=1,2$, denote the smooth mappings given by $h_1(x)=(x, y_0)$ ($x \in M_1$) and $h_2(y)=(x_0, y)$ ($y \in M_2$), respectively. Clearly, we have $h_i^*(A) \subset A_i$, $i=1, 2$. For any $f \in A$, we set

$$\tilde{f}=f-p_1^*(h_1^*(f))-p_2^*(h_2^*(f))+f(x_0, y_0).$$

Then $\tilde{f}$ lies in A and satisfies $\tilde{f}(x_0, y_0)=0$. It is not hard to verify that $d\tilde{f}$ vanishes at (x_0, y_0). It follows from Proposition 1.1 that $\tilde{f}$ vanishes identically on $M_1 \times M_2$. Hence,

$$f=p_1^*(h_1^*(f))+p_2^*(h_2^*(f))-f(x_0, y_0).$$

This formula means that

$$A=p_1^*(A_1)+p_2^*(A_2).$$

Since $p_1^*(A_1) \cap p_2^*(A_2)$ consists of all constant functions on $M_1 \times M_2$, it follows that $\dim A=\dim A_1+\dim A_2-1$. This proves Proposition 1.6.

§ 2. Parallel 1-forms and the first de Rham cohomology group

Let m be a connected Riemannian manifold with metric tensor g and ∇ the covariant differentiation of the Riemannian connection of M. Let f be any smooth function on M. We denote by H_f the Hessian of f defined by $H_f(X, Y)=(\nabla_X df)(Y)$ for all vector fields X and Y on M. Notice that H_f is a symmetric covariant 2-tensor on M.

LEMMA 2.1. *For any smooth function f on M and any smooth curve $c(t)$ in M, we have*

$$\frac{d^2}{dt^2} f(c(t))=H_f(\dot{c}(t), \dot{c}(t))+df\left(\frac{D\dot{c}(t)}{dt}\right).$$

Proof. For simplicity, let us denote by $F(t)$ the second derivative of $f(c(t))$ and set $H_f=H_f(\dot{c}(t), \dot{c}(t))$. We can assume that the curve $c(t)$ lies in a coordinate chart $(U, (y_1, \cdots, y_m))$ of M ($m=\dim M$). Let Γ_{ij}^k, $i, j, k=1, \cdots, m$, denote the components of the Riemannian connection Γ with respect to the coordinate system and set $c^i(t)=y_i \circ c(t)$, $i=1, \cdots, m$. Then we have

$$F(t)=\sum_{i,j=1}^{m}\frac{\partial^2 f}{\partial y_i \partial y_j}\cdot\frac{dc^i}{dt}\cdot\frac{dc^j}{dt}+\sum_{k=1}^{m}\frac{\partial f}{\partial y_k}\cdot\frac{d^2c^k}{dt^2}$$

and

$$H_f=\sum_{i,j=1}^{m}\left(\frac{\partial^2 f}{\partial y_i \partial y_j}-\sum_{k=1}^{m}\Gamma_{ij}^{k}\frac{\partial f}{\partial y_k}\right)\frac{dc^i}{dt}\frac{dc^j}{dt}.$$

Hence $F(t)-H_f$ is given by

$$\sum_{k=1}^{m}\frac{\partial f}{\partial y_k}\left(\frac{d^2c^k}{dt^2}+\sum_{i,j=1}^{m}\Gamma_{ij}^{k}\frac{dc^i}{dt}\frac{dc^j}{dt}\right),$$

which proves the formula.

PROPOSITION 2.1. *Let f be a smooth function on M. Then the following three statements are equivalent*:

1) *f is an affine function on M*;
2) *H_f vanishes identically on M*;
3) *df is a parallel 1-form on M.*

Proof. The equivalence of 2) and 3) is obvious. We shall prove the equivalence 1) $\Leftrightarrow$ 2). Suppose first 1). Let x be any point of M and let $c(t)$ be any geodesic with $c(0)=x$. From Lemma 2.1, we have

$$H_f(\dot{c}(0),\dot{c}(0))=\frac{d^2}{dt^2}f(c(t))\bigg|_{t=0}=0$$

and hence $H_f(u,u)=0$ for any $u\in T_x(M)$, where $T_x(M)$ denotes the tangent space to M at x. Since H_f is symmetric, we finally have $H_f(u,v)=0$ for all $u, v\in T_x(M)$. This implies 2). Now suppose 2). Then, from Lemma 2.1, we have

$$\frac{d^2}{dt^2}f(c(t))=H_f(\dot{c}(t),\dot{c}(t))=0$$

for any geodesic $c(t)$, which implies 1).

PROPOSITION 2.2. *Every affine function on M is a harmonic function.*

Proof. The Laplace-Beltrami operator Δ of M is given by $\Delta f=$Trace of H_f, f being a smooth function on M. Hence the assertion follows immediately from Proposition 2.1.

As an application of Proposition 2.1, we shall show the relation between the vector space $A(M,g)$ and the first de Rham cohomology group of M. Let $P^1(M,g)$ be the real vector space of all parallel 1-forms on M and $H^1(M)$ the first de Rham cohomology group of M.

THEOREM 2.3. *Let M be a connected Riemannian manifold with metric tensor g. Then there exist natural linear homomorphisms $j\colon A(M,g)\to P^1(M,g)$ and*

k: $P^1(M, g) \to H^1(M)$ such that the sequence

$$0 \longrightarrow \mathbf{R} \xrightarrow{i} A(M, g) \xrightarrow{j} P^1(M, g) \xrightarrow{k} H^1(M)$$

is exact. Hence,

$$0 \leqq \dim P^1(M, g) - a(M, g) \leqq \dim H^1(M).$$

In particular, if M is compact, then $\dim P^1(M, g) \leqq b_1(M)$. *Here $b_1(M)$ denotes the first Betti number of M.*

Proof. By Proposition 2.1, we can define the linear mapping j: $A(M, g) \to P^1(M, g)$ by $j(f) = df$ ($f \in A(M, g)$). Since every parallel 1-form ω on M is closed, it determines a cohomology class $k(\omega) \in H^1(M)$. Thus we get the linear mapping k: $P^1(M, g) \to H^1(M)$ and the sequence:

$$0 \longrightarrow \mathbf{R} \xrightarrow{i} A(M, g) \xrightarrow{j} P^1(M, g) \xrightarrow{k} H^1(M).$$

To prove the exactness of the sequence, it suffices to verify the relation $\operatorname{Ker} k \subset \operatorname{Im} j$. Let ω be any element of $\operatorname{Ker} k$. Then there is a smooth function f on M such that $\omega = df$. By Proposition 2.1, f belongs to $A(M, g)$ and hence $\omega = j(f) \in \operatorname{Im} j$. If M is compact, then every $f \in A(M, g)$ is constant (Proposition 1.2). This means that k: $P^1(M, g) \to H^1(M)$ is injective. Hence we get $\dim P^1(M, g) \leqq b_1(M)$.

Theorem 2.3 can be generalized to the case where M is an affinely connected manifold (see [2]).

§ 3. The main theorems

Let M be a connected Riemannian manifold with metric tensor g and $A(M, g)$ the real vector space of all affine functions on M. As before, we set $a(M, g) = \dim A(M, g) - 1$. For any smooth function f on M, we denote by $\operatorname{grad} f$ the gradient of f. Namely, $\operatorname{grad} f$ is a unique vector field on M such that $g(\operatorname{grad} f, X) = df(X)$ for any vector field X on M. Let $\mathbf{R}^n$ denote the n-dimensional Euclidean space with the standard metric ds^2. If $n = 0$, we regard $\mathbf{R}^0$ as a space consisting of only one point. Then we can state our main theorems as follows.

THEOREM 3.1. *Let M be an n-dimensional, connected and complete Riemannian manifold with metric tensor g. Then M is isometric to $\mathbf{R}^n$ if and only if $a(M, g) = n$.*

THEOREM 3.2. *Let M be a connected Riemannian manifold with metric tensor g and let $n = a(M, g)$. Assume that, for every $f \in A(M, g)$,* $\operatorname{grad} f$ *is a complete vector field on M. Then there exists uniquely (up to an isometry) a connected Riemannian manifold M' with metric tensor g' such that*

1) *M is isometric to the Riemannian product $\mathbf{R}^n \times M'$;*
2) *$a(M', g') = 0$.*

Moreover, the assumption is fulfilled if M is complete.

Remark. *The converse of Theorem* 3.2 *is also valid.* In fact, suppose 1) and 2) of Theorem 3.2. Then, from Propositions 1.4, 1.5 and 1.6, we have

$$a(M, g)=a(\mathbf{R}^n, ds^2)+a(M', g')=n\,.$$

Furthermore, every $f\in A(M, g)$ can be considered as an element of $A(\mathbf{R}^n, ds^2)$ (see the proof of Proposition 1.6). Hence we can see that $\operatorname{grad} f$ ($f\in A(M, g)$) is a complete vector field on M.

Now we shall prove our main theorems. Let M and g be as in Theorem 3.2.

LEMMA 3.1. *Let f be an affine function on M. Then*:
(1) $\operatorname{grad} f$ *is a parallel vector field on M*;
(2) $\operatorname{grad} f$ *is an infinitesimal isometry of M*;
(3) *Every integral curve of* $\operatorname{grad} f$ *is a geodesic of M.*

Proof. For simplicity, we write $X=\operatorname{grad} f$. (1): For all vector fields Y and Z on M, we have

$$\begin{aligned} g(\nabla_Y \operatorname{grad} f, Z) &= Yg(\operatorname{grad} f, Z)-g(\operatorname{grad} f, \nabla_Y Z)\\ &= Y(df(Z))-df(\nabla_Y Z)\\ &= H_f(Y, Z) \end{aligned}$$

and hence $\nabla_Y \operatorname{grad} f=0$ by Proposition 2.1. This proves (1).

(2): Let L_X denote the Lie differentiation with respect to X. Then, for all vector fields Y and Z on M, we have

$$\begin{aligned} (L_X g)(Y, Z) &= Xg(Y, Z)-g(L_X Y, Z)-g(Y, L_X Z)\\ &= g(\nabla_X Y-[X, Y], Z)+g(Y, \nabla_X Z-[X, Z])\\ &= g(\nabla_Y X, Z)+g(Y, \nabla_Z X)=0 \end{aligned}$$

and hence $L_X g=0$. This implies that X is an infinitesimal isometry of M.

(3) follows immediately from (1).

We shall first prove that *if M is complete then, for every $f\in A(M, g)$,* $\operatorname{grad} f$ *is a complete vector field on M* (see [3] p. 234). In fact, let $c(t)$ ($|t|<\varepsilon$, $\varepsilon>0$) be an integral curve of $\operatorname{grad} f$. Then, by Lemma 3.1(3), $c(t)$ is a geodesic. Since M is complete, $c(t)$ can be extended to a geodesic $x(t)$ defined for all $t\in\mathbf{R}$. Let I denote the subset of $\mathbf{R}$ consisting of all points t such that $\dot{x}(t)=(\operatorname{grad} f)_{x(t)}$. Clearly, I is non-empty and closed in $\mathbf{R}$. Let t_0 be any point of I and let $y(t)$ ($|t-t_0|<\varepsilon'$, $\varepsilon'>0$) be an integral curve of $\operatorname{grad} f$ with $x(t_0)=y(t_0)$. Then $y(t)$ is a geodesic with the initial condition $(x(t_0), (\operatorname{grad} f)_{x(t_0)})$. Hence $x(t)$ must coincide with $y(t)$ on a small open neighborhood of t_0. This shows that I is open in $\mathbf{R}$ and hence $I=\mathbf{R}$. Therefore every integral curve of $\operatorname{grad} f$ can be extended to an integral curve defined for all $t\in\mathbf{R}$. Hence $\operatorname{grad} f$ is complete.

Now we go back to the general situation of Theorem 3.2. Suppose that $n=$

$a(M, g)>0$. Let us fix a point x_0 of M.

LEMMA 3.2. *We can choose a basis* $1, f_1, \cdots, f_n$ *of* $A(M, g)$ *in such a way that*
1) $f_i(x_0)=0$, $i=1, \cdots, n$;
2) $g(\mathrm{grad}\, f_i, \mathrm{grad}\, f_j)=\delta_{ij}$, $i. j=1, \cdots, n$, *where* δ_{ij} *denotes Kronecker's delta.*

Proof. Let us fix a basis $1, f'_1, \cdots, f'_n$ of $A(M, g)$ and set

$$a_{ij}=g(\mathrm{grad}\, f'_i, \mathrm{grad}\, f'_j),\ i, j=1, \cdots, n .$$

From Lemma 3.1(1), we have $Xa_{ij}=0$, $i, j=1, \cdots, n$, for all vector fields X on M and hence a_{ij} is constant, $i, j=1, \cdots, n$. By Proposition 1.3, grad $f'_1, \cdots$, grad f'_n are linearly independent at each point of M. Hence the $n\times n$ symmetric matrix $A=(a_{ij})$ is positive definite. Consequently, there is a non-singular $n\times n$ matrix $B=(b_{ij})$ such that ${}^tBAB=I_n$, where ${}^tB$ denotes the transposed matrix of B and I_n the unit matrix of degree n. Now we set

$$f_i=\sum_{j=1}^{n} b_{ji}(f'_j-f'_j(x_0)), \quad i=1, \cdots, n .$$

Then $1, f_1, \cdots, f_n$ form a basis of $A(M, g)$ and satisfy $f_i(x_0)=0$ for all $i=1, \cdots, n$. Moreover, using the formular ${}^tBAB=I_n$, we can easily verify that $g(\mathrm{grad}\, f_i, \mathrm{grad}\, f_j)=\delta_{ij}$ for all $i, j=1, \cdots, n$. This completes the proof of Lemma 3.2.

From now on, we fix a basis $1, f_1, \cdots, f_n$ of $A(M, g)$ with the properties listed in Lemma 3.2. Let $p: M\rightarrow \mathbf{R}^n$ denote the smooth mapping given by $p(x)=(f_1(x), \cdots, f_n(x))$ $(x\in M)$. Then we have $p(x_0)=0$. Moreover, for every geodesic $c(t)$, there are two elements a and b such that $p(c(t))=at+b$ for any t where defined. Let $a=(a_1, \cdots, a_n)\in \mathbf{R}^n$ and let $X(a)$ denote the vector field on M given by

$$X(a)=\sum_{i=1}^{n} a_i\cdot \mathrm{grad}\, f_i .$$

Clearly, we have $X(a+b)=X(a)+X(b)$ and $X(sa)=sX(a)$ for all $a, b\in \mathbf{R}^n$ and $s\in \mathbf{R}$. Let $(x_1, \cdots, x_n)$ be the canonical coordinate system on $\mathbf{R}^n$ and $T_a(\mathbf{R}^n)$, $a\in \mathbf{R}^n$, the tangent space to $\mathbf{R}^n$ at a. As usual, we identify $T_a(\mathbf{R}^n)$ with $\mathbf{R}^n$ by the canonical absolute parallelism on $\mathbf{R}^n$.

LEMMA 3.3. *For any* $a\in \mathbf{R}^n$ *and any* $x\in M$, *we have* $p_*(X(a)_x)=a$, *where* p_* *denotes the differential of* p.

Proof. From the definition of p and Lemma 3.2, we have

$$\begin{aligned}(p_*(X(a)_x))(x_i)&=(df_i)\left(\sum_{j=1}^{n} a_j(\mathrm{grad}\, f_j)_x\right)\\ &=\sum_{j=1}^{n} a_j g((\mathrm{grad}\, f_i)_x, (\mathrm{grad}\, f_j)_x)=a_i\end{aligned}$$

for all $i=1, \cdots, n$, which proves Lemma 3.3.

LEMMA 3.4. *$p: M \to \mathbf{R}^n$ is a surjective submersion.*

Proof. It is clear from Proposition 1.3 that the rank of p is equal to n at each point of M. Let a be any point of $\mathbf{R}^n$ and let $c(t)$ be an integral curve of $X(a)$ with $c(0)=x_0$. Since $X(a)$ is complete, $c(t)$ is defined for all $t \in \mathbf{R}$. Moreover, by Lemma 3.1, $c(t)$ is a geodesic. Hence we can write $p(c(t))=bt$ for some $b \in \mathbf{R}^n$. From Lemma 3.3, we have $b=p_*(\dot{c}(0))=p_*(X(a)_{x_0})=a$ and hence $p(c(1))=a$. This shows that p is surjective.

Here we prove Theorem 3.1. Let m be an n-dimensional, connected and complete Riemannian manifold. Assume that $a(M, g)=n$. Then Lemma 3.4 implies that $p_*: T_x(M) \to \mathbf{R}^n$ is a linear isomorphism for any $x \in M$. By Lemmas 3.2 and 3.3, we can easily verify that $p^*ds^2=g$, where p^* denotes the codifferential of p. In other words, p is an isometric immersion of M onto $\mathbf{R}^n$. Therefore it suffices to prove that p is one-to-one. Let x and y be points of M satisfying $p(x)=p(y)$. Since M is complete, there exists a geodesic $c(t)$ $(t \in \mathbf{R})$ with $c(0)=x$ and $c(1)=y$. Then we can put $p(c(t))=at+b$ $(a, b \in \mathbf{R}^n)$. The condition $p(x)=p(y)$ now yields $a=0$ and hence $p_*(\dot{c}(0))=0$. Thus $\dot{c}(0)=0$, so y must coincide with x. Hence p is an isometry of M onto $\mathbf{R}^n$. The converse follows immediately from Propositions 1.4 and 1.5. This completes the proof of Theorem 3.1.

Let us return to the general case. By assumption, $X(a)$, $a \in \mathbf{R}^n$, is a complete vector field on M, so we can consider the 1-parameter family of diffeomorphisms F_t^a generated by $X(a)$. Let $F: \mathbf{R} \times \mathbf{R}^n \times M \to M$ denote the mapping given by $F(t, a, x)=F_t^a(x)$ $(t \in \mathbf{R},\ a \in \mathbf{R}^n,\ x \in M)$.

LEMMA 3.5. *$F: \mathbf{R} \times \mathbf{R}^n \times M \to M$ is a smooth mapping and satisfies*

$$p(F(t, a, x))=at+p(x)$$

for all $t \in \mathbf{R}$, $a \in \mathbf{R}^n$ and $x \in M$.

Proof. By Lemma 3.1(3), the curve $c(t)=F_t^a(x)$ is a geodesic with $\dot{c}(0)=X(a)_x$, so we can put $F(t, a, x)=\exp tX(a)_x$. Here exp denotes the exponential mapping of M. Let TM be the tangent bundle of M. We identify M with the zero section of TM. Then there is an open neighborhood N of M in TM with the properties: a) exp is defined on N, b) $\exp: N \to M$ is smooth and c) $tX(a)_x \in N$ for all $t \in \mathbf{R}$, $a \in \mathbf{R}^n$ and $x \in M$. If we set $F_0(t, a, x)=tX(a)_x$ $(t \in \mathbf{R},\ a \in \mathbf{R}^n,\ x \in M)$, then $F_0: \mathbf{R} \times \mathbf{R}^n \times M \to N$ satisfies $F=\exp \circ F_0$. Therefore it suffices to verify that F_0 is smooth. But this can be easily checked by taking suitable local coordinate systems. Now we can put $p(F(t, a, x))=bt+p(x)$ for some element $b=b(a, x)$ of $\mathbf{R}^n$. Differentiating this with respect to t at $t=0$, we obtain $p_*(X(a)_x)=b$ and hence $a=b$ by Lemma 3.3. This proves Lemma 3.5.

Let us set $M'=p^{-1}(0)$, 0 being the origin of $\mathbf{R}^n$. Then, by Lemma 3.4, M' is a closed submanifold of M. Let $j: M' \to M$ be the inclusion and g' the Riemannian metric on M' induced from that of M, i.e., $g'=j^*g$. Let $h: \mathbf{R}^n \times M' \to M$ and

$q: M \to M$ denote the smooth mappings defined by $h(a, x) = F(1, a, x)$ ($a \in \mathbf{R}^n$, $x \in M'$) and $q(x) = F(-1, p(x), x)$ ($x \in M$), respectively. From Lemma 3.5, we have $p(q(x)) = 0$ for any $x \in M$ and hence the image $q(M)$ is contained in M'. As M' is a closed submanifold of M, $q: M \to M'$ is smooth. Finally, for any $a \in \mathbf{R}^n$ and any $x \in M'$, let $h_a: M' \to M$ and $h_x: \mathbf{R}^n \to M$ denote the smooth mappings defined by $h_a(y) = h(a, y)$ ($y \in M'$) and $h_x(b) = h(b, x)$ ($b \in \mathbf{R}^n$), respectively. From Lemma 3.5, we have immediately the following lemma.

LEMMA 3.6. *For any $a \in \mathbf{R}^n$ and any $x \in M'$, we have $p \cdot h(a, x) = a$ and $q \circ h(a, x) = x$.*

It is not hard to see that h is a diffeomorphism of $\mathbf{R}^n \times M'$ onto M. In fact, the inverse mapping of h is given by the smooth mapping $p \times q$ of M onto $\mathbf{R}^n$, where $(p \times q)(x) = (p(x), q(x))$ for any $x \in M$.

LEMMA 3.7. *For all $a, b \in \mathbf{R}^n$ and all $s, t \in \mathbf{R}$, we have $F_t^{a+b} = F_t^a \circ F_t^b$ and $F_t^{sa} = F_{st}^a$.*

Proof. By Lemma 3.1(1), $X(a)$, $a \in \mathbf{R}^n$, is a parallel vector field. Hence,

$$[X(a), X(b)] = \nabla_{X(a)} X(b) - \nabla_{X(b)} X(a) = 0 \qquad (a, b \in \mathbf{R}^n).$$

Now the first formula follows from this fact and the formula:

$$X(a+b) = X(a) + X(b).$$

The second formula may be obvious.

LEMMA 3.8. *For all $x \in M'$, $a \in \mathbf{R}^n$ and $u \in T_a(\mathbf{R}^n)$, we have $(h_x)_*(u) = X(u)_y$, where $y = h(a, x)$.*

Proof. Let $b = p(y)$; then $p(F_t^a(y)) = at + b$. From Lemma 3.7, we have

$$\begin{aligned} q(F_t^a(y)) &= F_{-1}^{at+b} \circ F_t^a(y) = F_{-1}^b \circ F_{-1}^{at} \circ F_t^a(y) \\ &= F_{-1}^b \circ F_{-t}^a \circ F_t^a(y) = F_{-1}^b(y) \end{aligned}$$

and hence $q(F_t^a(y)) = q(y)$ for any $t \in \mathbf{R}$ and any $a \in \mathbf{R}^n$. This yields $q_*(X(u)_y) = 0$. By Lemma 3.6, $q \circ h_x: \mathbf{R}^n \to M'$ is a constant mapping, so $q_*((h_x)_*(u)) = 0$ for any $u \in T_a(\mathbf{R}^n)$. Thus we get $q_*((h_x)_*(u) - X(u)_y) = 0$. On the other hand, by Lemma 3.6, $p \circ h_x: \mathbf{R}^n \to \mathbf{R}^n$ is the identity mapping, so $p_*((h_x)_*(u)) = u$. This formula together with Lemma 3.3 yields $p_*((h_x)_*(u) - X(u)_y) = 0$. Since $p \times q$ is a diffeomorphism of M onto $\mathbf{R}^n \times M'$, it follows that $(h_x)_*(u) = X(u)_y$. This proves Lemma 3.8.

We are now in a position to prove that h is an isometry. Let (a, x) be any point of $\mathbf{R}^n \times M'$ ($a \in \mathbf{R}^n$, $x \in M'$). Notice that the tangent space to $\mathbf{R}^n \times M'$ at (a, x) is isomorphic to the direct sum $T_a(\mathbf{R}^n) \oplus T_x(M')$ by the mapping $(h_x)_* + (h_a)_*$. Therefore it suffices to prove the following lemma.

LEMMA 3.9. *For all $u, v \in T_a(\mathbf{R}^n)$ and all $X, Y \in T_x(M')$, we have*

1) $g((h_x)_*(u), (h_x)_*(v)) = ds^2(u, v)$,

2) $g((h_x)_*(u), (h_a)_*(X))=0$,
3) $g((h_a)_*(X), (h_a)_*(Y))=g'(X, Y)$.

Proof. Let $\mathbf{e}_1, \cdots, \mathbf{e}_n$ be the standard orthonormal basis of $\mathbf{R}^n$ (and hence of $T_a(\mathbf{R}^n)$). From Lemma 3.8, we have $(h_x)_*(\mathbf{e}_i)=X(\mathbf{e}_i)_y=(\operatorname{grad} f_i)_y$ $(y=h(a, x))$ and hence $g((h_x)_*(\mathbf{e}_i), (h_x)_*(\mathbf{e}_j))=\delta_{ij}$, $i, j=1, \cdots, n$, by Lemma 3.2. This proves 1). By Lemma 3.6, $p \circ h_a: M' \to \mathbf{R}^n$ is a constant mapping, so $p_* \circ (h_a)_*(X)=0$. Hence we have

$$\begin{aligned} g((h_x)_*(\mathbf{e}_i), (h_a)_*(X)) &= g((\operatorname{grad} f_i)_y, (h_a)_*(X)) \\ &= (df_i)_y((h_a)_*(X)) \\ &= (p_* \circ (h_a)_*(X))(x_i)=0 \end{aligned}$$

for all $i=1, \cdots, n$, where $(x_1, \cdots, x_n)$ denotes the canonical coordinate system on $\mathbf{R}^n$. We have thereby prove 2). Since $h_a(x')=F_1^a(x')$ for any $x' \in M'$, we can write $h_a=F_1^a \circ j$. By Lemma 3.1(2), $F_1^a: M \to M$ is an isometry of M. Hence we have

$$h_a^* g=j^*(F_1^a)^* g=j^* g=g',$$

which proves 3).

The assertion 2) of Theorem 3.2 follows easily from Proposition 1.6.

We now proceed to show the uniqueness of the decomposition. Suppose that there are a connected Riemannian manifold $\bar{M}$ with metric tensor $\bar{g}$ and an isometry s of $\mathbf{R}^n \times \bar{M}$ onto M. We shall prove that $\bar{M}$ is isometric to M'. For simplicity, we write $\tilde{g}=ds^2+\bar{g}$ and $A(\mathbf{R}^n \times \bar{M})=A(\mathbf{R}^n \times \bar{M}, \tilde{g})$.

LEMMA 3.10. *There exists a Euclidean motion T of $\mathbf{R}^n$ such that $p \circ s(a, z)=T(a)$ for any $a \in \mathbf{R}^n$ and any $z \in \bar{M}$.*

Proof. The canonical coordinate function x_i on $\mathbf{R}^n$ can be regarded as a function on $\mathbf{R}^n \times \bar{M}$. Then $1, x_1, \cdots, x_n$ form a basis of $A(\mathbf{R}^n \times \bar{M})$ and satisfy $\tilde{g}(\operatorname{grad} x_i, \operatorname{grad} x_j)=\delta_{ij}$, $i, j=1, \cdots, n$. We set $f_i'=s^* f_i$, $i=1, \cdots, n$. Since s is an isometry, $1, f_1', \cdots, f_n'$ form a basis of $A(\mathbf{R}^n \times \bar{M})$ and satisfy $\tilde{g}(\operatorname{grad} f_i', \operatorname{grad} f_j')=\delta_{ij}$, $i, j=1, \cdots, n$. Hence there are an orthogonal $n \times n$ matrix (a_{ij}) and an element $(b_1, \cdots, b_n)$ of $\mathbf{R}^n$ such that

$$f_i'=\sum_{j=1}^{n} a_{ji} x_j+b_i \quad (i=1, \cdots, n).$$

Since $p \circ s=(f_1', \cdots, f_n')$, we have only to set $x_i \circ T=f_i'$, $i=1, \cdots, n$. This proves Lemma 3.10.

Now we set $a=T^{-1}(0) \in \mathbf{R}^n$ and $s_a(z)=s(a, z)$ $(z \in \bar{M})$. Then Lemma 3.10 implies that the image $s_a(\bar{M})$ is contained in M'. We shall first prove that s_a is a diffeomorphism of $\bar{M}$ onto M'. In fact, let $\pi: \mathbf{R}^n \times \bar{M} \to \bar{M}$ be the natural projection and let $r_a: M' \to \bar{M}$ denote the smooth mapping given by $r_a(x)=\pi \circ s^{-1}(x)$ $(x \in M')$. It is easy to see that $r_a \circ s_a(z)=z$ for any $z \in \bar{M}$. For any $x \in M'$, we set $(b, z)=s^{-1}(x)$

($b \in \mathbf{R}^n$, $z \in \bar{M}$). Then we have $p \circ s(b, z) = p(x) = 0$ and hence $T(b) = 0$ by Lemma 3.10, so $a = b$. Now we can verify that $s_a \circ r_a(x) = x$ for any $x \in M'$. Hence s_a is a diffeomorphism. Let $k: \bar{M} \to \mathbf{R}^n \times \bar{M}$ denote the inclusion given by $k(z) = (a, z)$ ($z \in \bar{M}$). Then we have $j \circ s_a = s \circ k$ and hence

$$s_a^* g' = s_a^*(j^* g) = k^*(s^* g) = k^* \tilde{g} = \bar{g} .$$

Therefore s_a is an isometry of $\bar{M}$ onto M'. We have thus proved Theorem 3.2.

Added in proof. After completing this article, the author was informed by Dr. N. Innami that he proved from the quite different point of view the analogous result to Theorems 3.1 and 3.2 for complete Riemannian manifolds (N. Innami, Splitting theorems of Riemannian manifolds, Composito Math. vol. 47 (1982), 237–247). Then our main theorem can be considered as a refinement of Innami's theorem. His proof depends strongly on the theory of geodesics of Riemannian manifolds. On the contrary, our proof may be elementary and can be applicable to affinely connected manifolds (see [2]).

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